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Generalized Axisymmetric Potentials*

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1. INTRODUCTION

The value distribution of generalized axisymmetric potentials (GASP) in E^N, of complex order, is characterized geometrically in terms of the range of associated analytic functions of one complex variable by drawing together methods from the theory of polynomials of one complex variable and the integral operator methods of Bergman and Gilbert. Comparisons are made between the ranges of GASP of different orders that have the same associate; a connection with Newtonian Potentials is established.

Recent studies by Marden [3, 4] used integral operator methods to bound the zero sets of harmonic polynomials and infrapolynomials in E^N. The zeros of associated polynomials of one complex variable locate cones in E^N where axisymmetric harmonic polynomials have no zeros.

The author [7–9] considered the distribution of values of axisymmetric potentials and potentials in three real variables by using associated analytic functions of one complex variable to determine open convex sectors consisting of values excluded by these potentials. The present study establishes closed sets that bound the range of the GASP, depend directly on its associate, and are in general, neither convex nor unbounded.

A generalized axisymmetric potential of order μ, H_μ, satisfies the singular partial differential equation

$$\frac{\partial^2 H_\mu}{\partial x^2} + \frac{\partial^2 H_\mu}{\partial \rho^2} + \frac{2\mu}{\rho} \frac{\partial H_\mu}{\partial \rho} = 0 \tag{1}$$

for $\mathrm{Re}(\mu) > 0$. When $2\mu = N - 2$, the relation

$$x = x_1, \qquad \rho^2 = x_2{}^2 + \cdots + x_N{}^2 \tag{2}$$

holds between the Cartesian coordinates $(x_1, ..., x_N) \in E^N$ and the axisymmetric coordinates x and ρ. Then Eq. (1) becomes the axially symmetric LaPlace's equation in E^N. Axisymmetric potentials arise when $\mu = \frac{1}{2}$.

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In developing his theory of the singularities of GASP, Gilbert [2, p. 168] uses the integral representation for Gegenbauer polynomials

$$r^n C_n^{\mu}(\cos\theta) = \frac{2^{1/2}\Gamma(n+2\mu)}{n!\,\Gamma(\mu)^2}\int_0^{\pi}(x+i\rho\cos t)^n(\sin t)^{2\mu-1}\,dt \tag{3}$$

and the polar coordinates $x = r\cos\theta$, $\rho = r\sin\theta$ to *associate* with each GASP

$$\tilde{H}_{\mu}(r,\theta) = H_{\mu}(x,\rho) = \sum_{n=0}^{\infty}\frac{a_n n!\, r^n}{\Gamma(n+2\mu)}C_n^{\mu}(\cos\theta), \qquad a_n\in\mathbb{C} \tag{4}$$

an analytic function

$$h(\zeta) = \sum_{n=0}^{\infty} a_n\zeta^n \tag{5}$$

by means of an operator equivalent to the operator we refer to as $H_{\mu} \equiv A_{\mu}(h)$ where

$$H_{\mu}(x,\rho) = \alpha_{\mu}\int_0^{\pi} h(x+i\rho\cos t)\,dm_{\mu}(t), \tag{6}$$

$\alpha_{\mu} = 2^{1-2\mu}/\Gamma(\mu)^2$ with measure $dm_{\mu}(t) = (\sin t)^{2\mu-1}\,dt$. The principal values of the weight $(\sin t)^{2\mu-1}$, $2^{2\mu}$ and $\Gamma(\mu)$ are taken when μ is complex and

$$\beta_{\mu} = \int_0^{\pi} dm_{\mu}(t) = \pi^{1/2}\Gamma(\mu)/\Gamma(\mu+\tfrac{1}{2}). \tag{7}$$

Legendre's duplication formula evaluates the product $\alpha_{\mu}\beta_{\mu} = (\Gamma(2\mu))^{-1}$.

We consider the domain of the associate h to be an *axiconvex* set [2, p. 138] $\omega\subset\mathbb{C}$ containing the origin with the property that $\zeta t + \bar{\zeta}(1-t)\in\omega$, $0\leqslant t\leqslant 1$, whenever $\zeta\in\omega$. The corresponding domain of the H_{μ} is the *axisymmetric* set $\Omega = \{(x,\rho)\mid x+i\rho\in\omega\}$ which may be viewed, when $2\mu = N-2$, by rotating ω about the axis of symmetry in E^N.

The family of GASP generated from (6) on Ω by fixing h and ω and varying μ is specified by

$$\mathscr{C}(h) = \{H_{\mu}\mid H_{\mu} = A_{\mu}(h), \operatorname{Re}(\mu) > 0\}. \tag{8}$$

Restricting the order μ so that $\operatorname{Im}(\mu) = 0$ reduces $\mathscr{C}(h)$ to the subfamily $\mathscr{R}(h)$.

For each circle $X_0 : x = x_0$, $\rho = \rho_0$ in Ω there corresponds a segment

$$\sigma_0 = \sigma(X_0) = \{x_0 + i\rho_0\cos t \mid 0\leqslant t\leqslant\pi\} \tag{9}$$

in ω and the image of σ_0 under h, $h(\sigma_0)$. The closed convex hull of the curve $h(\sigma_0)$, $h^*(\sigma_0)$, and the set

$$h^*(\omega) = \{h^*(\sigma_0) \mid \sigma_0 \subset \omega\} \tag{10}$$

are of prime interest in studying the values of H_μ. In general, $h^*(\omega)$ is not convex (see Example 1) but is contained in the closed convex hull of $h(\omega)$. If necessary, ω may be restricted so that $h^*(\omega)$ does not degenerate to $\mathbb{C}$.

2. Value Distribution of GASP Having Real Order

We begin our study of GASP having real order in

THEOREM 1. *Let $\mathscr{R}(h)$ be a family of* GASP *defined on Ω. Then on Ω, each $H_\mu \in \mathscr{R}(h)$ may be represented as*

$$H_\mu(X) = G_\mu(X)/\Gamma(2\mu), \qquad X \in \Omega \tag{11}$$

where G_μ assumes its values in the set $h^*(\omega)$ for all orders μ. In particular, if $h^*(\omega) \not\ni 0$, then H_μ has no zeros on Ω.

Proof. For each $H_\mu \in \mathscr{R}(h)$, we define the GASP G_μ by

$$G_\mu(X_0) = (\alpha_\mu \beta_\mu)^{-1} H_\mu(X_0), \qquad X_0 \in \Omega \tag{12}$$

and then transform this representation into the equivalent integral

$$\int_0^\pi v(t)\, dm_\mu(t) = 0, \qquad v(t) \equiv [h(x_0 + i\rho_0 \cos t) - G_\mu(X_0)] \tag{13}$$

by means of the operator A_μ.

If there is a circle $X_0 \in \Omega$ for which $G_\mu(X_0) \notin h^*(\omega)$, then $G_\mu(X_0) \notin h^*(\sigma_0)$. So that for some real constant δ, the inequality

$$\delta < \arg v(t) < \delta + \pi \tag{14}$$

holds on $0 < t < \pi$. The integrand of (13) must satisfy these same bounds

$$\delta < \arg[v(t)\, dm_\mu(t)] < \delta + \pi \tag{15}$$

for $0 < t < \pi$ due to the positivity of the measure $dm_\mu(t)$.

Consequently, the integral in (13) viewed as the limit of a sum of vectors drawn from the origin to points in the half-plane $\delta < \arg \zeta < \delta + \pi$ cannot vanish. Thus, Eq. (13) cannot be satisfied, so that $G_\mu(X_0) \neq \Gamma(2\mu) H_\mu(X_0)$, which contradicts the definition of G_μ and proves the theorem.

Using this result to measure the separation in values of GASP induced by a separation of values of the associates, we consider a fixed circle $X_0 : x = x_0$, $\rho = \rho_0$ and all circles $X : x = x_1$, $\rho = \rho_1$ for which $x_1 = x_0$ and $\rho_1 \leqslant \rho_0$. This relation is indicated by $X \prec X_0$. Then Theorem 1 leads us to

COROLLARY 1.1. *Let $\mathscr{R}(h)$ and $\mathscr{R}(k)$ have Ω as their domain. Let X_0 and Y_0 be circles in Ω for which $h^*(\sigma(X_0)) \cap k^*(\sigma(Y_0)) = \varnothing$. Then for all $H_\mu \in \mathscr{R}(h)$, all $K_\nu \in \mathscr{R}(k)$ and all circles $X \prec X_0$ and $Y \prec Y_0$*

$$\Gamma(2\mu) H_\mu(X) \neq \Gamma(2\nu) H_\nu(Y). \tag{16}$$

Thus, the only possible zeros of H_μ are on those circles X_0 for which $h^*(\sigma_0) \ni 0$.

Proof. Let us assume that under the hypothesis, Ω contains circles X_0 and Y_0; $\mathscr{R}(h)$ and $\mathscr{R}(k)$ contain GASP H_μ and K_ν for which equality holds in (16). By Theorem 1, H_μ and H_ν may be represented as

$$H_\mu(X_0) = G_\mu(X_0)/\Gamma(2\mu) \qquad \text{and} \qquad K_\nu(Y_0) = J_\nu(Y_0)/\Gamma(2\nu) \tag{17}$$

with $G_\mu(X_0) \in h^*(\sigma(X_0))$ and $J_\nu(Y_0) \in k^*(\sigma(Y_0))$. The assumption of equality in (16) yields

$$G_\mu(X_0) = J_\nu(Y_0) \tag{18}$$

a violation of the null intersection of $h^*(\sigma(X_0))$ and $k^*(\sigma(Y_0))$. In particular, if $k \equiv 0$ and $0 \notin h^*(\sigma(X_0))$, then $H_\mu(X) \neq 0$ for all $\mu > 0$ and all circles $X \prec X_0$, which establishes the result.

Let $\{h_n\}_{n=1}^m$ be a set of functions analytic on the domain ω and $\{\lambda_n\}_{n=1}^m$ be complex constants with h defined by the linear combination

$$h(\zeta) = \sum_{n=1}^{m} \lambda_n h_n(\zeta), \qquad \zeta \in \omega. \tag{19}$$

Then the observation [1, p. 415] that the relation

$$h^*(\sigma_0) = \sum_{n=1}^{m} \lambda_n h_n^*(\sigma_0) \tag{20}$$

holds for each segment $\sigma_0 \subset \omega$ establishes a connection between the value distributions of families of GASP analogous to those due to Marden, Walsh [5, p. 77 ff.], and others on null sets of linear combinations of polynomials of one complex variable. We find

COROLLARY 1.2. *Let $\{\mathscr{R}(h_n)\}_{n=1}^m$ have Ω as their common domain. Let each $H_{\mu_n} \in \mathscr{R}(h_n)$ have order μ for $1 \leqslant n \leqslant m$. Then for any set of complex constants $\{\lambda_n\}_{n=1}^m$, these* GASP *are connected by*

$$\sum_{n=1}^{m} \lambda_n H_{\mu_n}(X_0) = (\Gamma(2\mu))^{-1} \sum_{n=1}^{m} \lambda_n \eta_n(X_0), \qquad X_0 \in \Omega \tag{21}$$

with $\eta_n(X) \in h_n{}^*(\sigma_0)$ for $X \prec X_0$, $1 \leqslant n \leqslant m$.

These results extend the previously mentioned work by the author and Marden whenever σ_0 is a segment in the domain of h and ξ is a complex constant for which the curve $h(\sigma_0) - \xi$ lies in a convex sector through the origin. We list this application in the form found in

THEOREM 2. *Let h be an analytic function on ω and $\sigma_0 \subset \omega$. Let ξ be a complex constant such that*

$$\xi + \lambda \notin h(\sigma_0) \tag{22}$$

for all λ in the complement of the convex sector $|\arg(\zeta - \xi)| < \gamma$. Then for all circles $X \prec X_0$ and all $H_\mu \in \mathscr{R}(h)$,

$$H_\mu(X) \neq (\xi + \lambda)/\Gamma(2\mu) \tag{23}$$

for all λ in the sector $|\arg(\zeta - \xi)| > \gamma$.

Marden [3, p. 140], recognizing that the zeros of the polynomial

$$h(\zeta) - \xi = \sum_{n=0}^{m} a_n \zeta^n - \xi, \qquad a_m \neq 0 \tag{24}$$

lie in the disk

$$|\zeta| \leqslant c = 1 + \max\{|a_0 - \xi|/|a_m|, |a_1/a_m|, \ldots, |a_{m-1}/a_m|\} \tag{25}$$

determines cones [3, p. 140] in E^3 in which harmonic polynomials omit the real value ξ. We shall refer to the solutions of

$$0 < \rho \leqslant \pm\, x \tan(\pi/2m) - c \sec(\pi/2m) \tag{26}$$

as *cones* where x and ρ are the axisymmetric coordinates of Eq. (1).

Inasmuch as Luca's theorem [5, p. 22] states that all the critical points of a nonconstant polynomial lie in the closed convex hull of its zeros, the zeros of each polynomial

$$(d/d\zeta)^p\, [h(\zeta) - \xi] = \sum_{n=0}^{m-p} \frac{a_{n+p}(n+p)!}{n!} \zeta^n \tag{27}$$

$1 \leqslant p \leqslant m - 1$ satisfy (25). The reasoning of theorem 2 produces

COROLLARY 2.1. *Let ξ be an arbitrary real constant and $H_\mu{}^p$ be the generalized axisymmetric harmonic polynomials*

$$\tilde{H}_\mu{}^p(r, \theta) = \sum_{n=0}^{m-p} \frac{a_{n+p}\Gamma(n+p)\, r^n}{\Gamma(n+2\mu)} C_n{}^\mu(\cos\theta) \tag{28}$$

for $0 \leqslant p \leqslant m-1$. *Then on each circle* X_0 *in the cones* (25),

$$H_\mu{}^p(X_0) \neq \xi[1 - \operatorname{sgn} p]/\Gamma(2\mu), \qquad \mu > 0. \tag{29}$$

An application of Theorem 1 determines the value distribution of a family of GASP that is periodic in the axial coordinate.

EXAMPLE 1. Let us consider the GASP

$$\tilde{H}_\mu(r, \theta) = \sum_{n=0}^{\infty} \frac{a^n i^n r^n}{\Gamma(n+2\mu)} C_n{}^\mu(\cos\theta), \qquad \mu > 0 \tag{30}$$

for real nonzero values of a. The cylinder $\Omega(b) = \{(x, \rho) \mid 0 \leqslant \rho \leqslant b\}$ is defined for $b > 0$. Then for all circles X_0 in $\Omega(b)$ and all $\mu > 0$:

$$H_\mu(X_0) \neq \xi/\Gamma(2\mu) \tag{31}$$

for all ξ in the complement of the annulus

$$A(a, b) = \{\zeta \in \mathbb{C} \mid \exp(-|a|b) \leqslant |\zeta| \leqslant \exp(|a|b)\}. \tag{32}$$

Consequently, no H_μ has a finite zero.

Proof. The H_μ are entire functions and periodic with

$$H_\mu(x + 2\pi/a, \rho) = H_\mu(x, \rho) \tag{33}$$

because their associate is identified by (5) as the exponential $h(\zeta) = \exp(ia\zeta)$. Since $h(\sigma_0) = \{\zeta \in \mathbb{C} \mid \exp(-|a|\rho_0) \leqslant |\zeta| \leqslant \exp(|a|\rho_0),\ \arg\zeta = ax_0\}$, the associate maps the infinite strip $|\operatorname{Im}\zeta| \leqslant b$ onto the annulus $A(a, b)$. The relation $h^*(\sigma_0) = h(\sigma_0)$ holds for this associate so that $h^*(\omega) = h(\omega) = A(a, b)$. We conclude by Theorem 1 that $\Gamma(2\mu)\, H_\mu(X_0) \in A(a, b)$ if $X_0 \in \Omega(b)$. Since $A(a, b) \not\ni 0$ for $b < \infty$, there is no circle X within a finite distance of the origin on which H_μ vanishes.

Let us turn our attention to the influence of the order of a GASP on its values. For each fixed order ν, we specify the values of GASP of order $\mu > \nu$ in terms of those of order ν. The corresponding relations for orders $\mu < \nu$ can then be found by reflection in the unit circle $|\zeta| = 1$.

For these comparisons we shall deal with associates which are *convex* on their domains. Convexity implies that if h is defined on ω and $\sigma_0 \subset \omega$, then

$\gamma_h(\sigma_0)$, the angle subtended by the curve $h(\sigma_0)$ at the origin, does not exceed a fixed positive constant $\gamma_h < \pi$.

These comparisons will require the configuration $E(\gamma)$ having symmetry in the real axis which consists of the union of the two disks

$$D(\gamma) = \{\zeta \in \mathbb{C} \mid 2 \mid \zeta \mid \mid \zeta - 1 \mid \cos(\pi - \gamma) \geqslant \mid \zeta \mid^2 + \mid \zeta - 1 \mid^2 - 1\} \quad (34)$$

(for diagram see [5; p. 31]) formed from the set of points for which the segment [0, 1] subtends an angle of at least $\pi - \gamma$ and the region

$$S(\gamma) = \{\zeta \in \mathbb{C} \mid \mid \arg \zeta \mid \leqslant \gamma \leqslant \mid \arg(\zeta - 1)\mid\} \quad (35)$$

included in the sector $|\arg \zeta| \leqslant \gamma$ and exterior to the sector $|\arg(\zeta - 1)| < \gamma$. We consider

THEOREM 3. *Let $\mathscr{R}(h)$ be convex on Ω and $H_\nu \in \mathscr{R}(h)$. Then for each $H_\mu \in \mathscr{R}(h)$ with order $\mu > \nu$, there exists a function η such that ηH_ν is a nonvanishing* GASP *of order μ with H_μ and H_ν connected by*

$$\alpha_\nu H_\mu(X) = \alpha_\mu \eta(X) H_\nu(X), \qquad X \in \Omega \quad (36)$$

where $\eta(X) \in E(\gamma_h)$.

Proof. Inasmuch as H_μ and H_ν can have no zeros on Ω, Eq. (36) may be used to define a continuous nonvanishing function η on Ω. The operator A_μ converts Eq. (36) into the equivalent form

$$\int_0^\pi h(x_0 + i\rho_0 \cos t)(\sin t)^{2\nu-1+j} \Delta(\mu, j, t)\, dt = 0 \quad (37)$$

where

$$\Delta(\mu, j, t) = [(\sin t)^{2(\mu-\nu)-j} - \eta(X_0)(\sin t)^{-j}], \quad (38)$$

$0 < t < \pi$. If $\eta(X_0)$ is a value for which

$$\delta < \arg \Delta(\mu, j, t) < \delta + \pi - \gamma, \qquad 0 < t < \pi \quad (39)$$

for some constant δ, then Eq. (36) is contradicted. For then the combination of (39) and the convexity of $\mathscr{R}(h)$ on Ω,

$$\kappa < \arg[h(x_0 + i\rho_0 \cos t)(\sin t)^{2\nu-1+j}] < \kappa + \gamma \quad (40)$$

for some κ results in the bound

$$\kappa + \delta < \arg[h(x_0 + i\rho_0 \cos t)(\sin t)^{2\nu-1+j} \Delta(\mu, j, t)] < \pi + \kappa + \delta \quad (41)$$

on $0 < t < \pi$, which according to previous reasoning leads to a contradiction of (36). Therefore, our problem reduces to locating the complements of sets in $\mathbb{C}$ that violate (39).

If $|\arg \eta(X_0)| > \gamma$, set $j = \mu - \nu$ to obtain (41). If $|\arg(\eta(X_0) - 1)| < \gamma$, or if $\eta(X_0) \notin D(\gamma)$, set $j = 0$ to produce (41), which completes the proof.

When considering linear combinations of GASP, this result may be recast in a form that is independent of the convexity of a specific family of GASP by suitably restricting the domain ω. Such a reformulation is found in

COROLLARY 3.1. *Let $\{\mathscr{R}(h_n)\}_{n=1}^{m}$ have compact domain Ω_* and $\{\lambda_n\}_{n=1}^{m+1}$ be a choice of complex constants for which the set $\sum_{n=1}^{m} \lambda_n h_n(\omega_*)$ is in a convex sector $|\arg(\zeta - \lambda_{m+1})| < \gamma$. Then the members H_{μ_n} and H_{ν_n} of $\mathscr{R}(h_n)$ with orders $\mu_n = \mu > \nu = \nu_n$ for $1 \leqslant n \leqslant m$ are connected by the equation*

$$\alpha_\nu \sum_{n=1}^{m} \lambda_n H_{\mu_n}(X) = \alpha_\mu \eta_\lambda(X) \sum_{n=1}^{m} \lambda_n H_{\nu_n}(X) + \Lambda, \qquad X \in \Omega_* \tag{42}$$

where $\Lambda = \alpha_\mu \alpha_\nu(\lambda_{m+1} - \lambda)(\beta_\mu - \beta_\nu \eta_\lambda(X))$ with $\eta_\lambda(X) \in E(\gamma)$ whenever λ is the sector $|\arg(\lambda_{m+1} - \zeta)| < \gamma$ or $\lambda = 0$.

Applying the triangle inequality to this theorem and using the fact that the set $D(\gamma)$ lies in the disk $|\zeta - 1/2| \leqslant \csc((\pi - \gamma)/2)$ produces bounds on the magnitudes of GASP drawn from the same family as a function of their order. These estimates do not appear to be obtainable by other methods.

COROLLARY 3.2. *Let $\mathscr{R}(h)$ be convex on Ω. Then for each circle $X_0 \in \Omega$ and all orders $\mu > \nu$,*

$$|H_\mu(X_0)/H_\nu(X_0)| \leqslant 4^{\mu-\nu}(\Gamma(\mu)/\Gamma(\nu))^2 \csc((\pi - \gamma_h)/2). \tag{43}$$

3. AN EXTENSION TO GASP HAVING COMPLEX ORDER

The methods of the previous section are adopted to obtain a geometric connection between the value distribution of GASP of real and complex order in

THEOREM 4. *Let $\mathscr{C}(h)$ be convex on Ω. Then each $H_\delta \in \mathscr{C}(h)$ may be represented as*

$$\alpha_\mu H_\delta(X) = \alpha_\delta \eta(X) H_\mu(X), \qquad X \in \Omega \tag{44}$$

with $H_\mu \in \mathscr{R}(h)$ and $|\eta(X)| \leqslant \csc((\pi - \gamma_h)/2)$ whenever $\operatorname{Re} \delta \geqslant \mu$.

Proof. The function η may be defined by (44) since the GASP, H_μ has no zeros on Ω. Let us rewrite (44) as

$$\int_0^\pi h(x_0 + i\rho_0 \cos t)(\sin t)^{2\mu-1} [(\sin t)^{2\beta} - \eta(X_0)]\, dt = 0 \qquad (45)$$

where $X_0 \in \Omega$ and $\beta = \delta - \mu$. Then the envelope of the vector $w(t) = (\sin t)^{2\beta}$, $0 < t < \pi$, is in the disk $|\zeta| \leqslant 1$ for $\operatorname{Re} \beta \geqslant 0$. Thus, if

$$|\eta(X_0)| > \csc((\pi - \gamma_h)/2),$$

we reason as in the previous theorems to find that Eq. (44) is contradicted.

Theorems 1 and 4 in combination reveal

COROLLARY 4.1. *Let $\mathscr{C}(h)$ be convex on Ω. Then on Ω, $H_\delta \in \mathscr{C}(h)$ has the form*

$$H_\delta(X) = \alpha_\delta \beta_\mu \eta(X)\, G_\mu(X), \qquad X \in \Omega \qquad (46)$$

with $G_\mu(X) \in h^*(\omega)$ and $|\eta(X)| \leqslant \csc((\pi - \gamma_h)/2)$ whenever $\operatorname{Re} \delta \geqslant \mu$.

These results are applied in

EXAMPLE 2. On each cylinder $\Omega(b)$, the GASP

$$\tilde{H}_\delta(r, \theta) = \sum_{n=0}^{\infty} \frac{a^n i^n r^n}{\Gamma(n + 2\delta)} C_n{}^\delta(\cos \theta) \qquad \operatorname{Re} \delta > 0 \qquad (47)$$

omits all values of the form $\alpha_\delta \beta_{\mu_0} \xi$ for all $|\xi| > \exp(|a| b)$ and all $\operatorname{Re} \delta \geqslant \mu_0 > 0$, μ_0 fixed.

4. A CLASS OF NEWTONIAN POTENTIALS

Here we shall use an operator introduced by Marden [3] to generate axisymmetric potentials. With this operator, an axisymmetric potential $H \equiv H_{1/2}$ having as its domain $\Omega \subset E^3$ may be represented as

$$H(X) = (1/2\pi i) \int_{\mathscr{L}_1} h(u)\, \zeta^{-1}\, d\zeta, \qquad u = x + (i\rho/2)(\zeta + \zeta^{-1}) \qquad (48)$$

where h is analytic on Ω and $\mathscr{L}_1$ symbolizes one circuit around $|\zeta| = 1$ in the positive direction. Cauchy's formula transforms this integral into

$$\int_{\mathscr{L}_1} h(u)\, \zeta^{-1}\, d\zeta = 1/2\pi i \int_{\mathscr{L}_1} \int_{\mathscr{L}_2} \frac{h(s)\, ds\, d\zeta}{\zeta[u - s]} \qquad (49)$$

where $\mathscr{L}_2$ is a contour, a simple closed smooth rectifiable curve, containing Ω on which h is analytic and for which $\mathscr{L}_2 \cap \partial\Omega = \phi$. The double integral is absolutely integrable, permitting the interchange of orders of integration which gives

$$\int_{\mathscr{L}_1} h(u)\, \zeta^{-1}\, d\zeta = \int_{\mathscr{L}_2} \frac{h(s)\, ds}{\{(x-s)^2 + \rho^2\}^{1/2}} \tag{50}$$

where the branch of the radical is chosen so that $\{x^2 + \rho^2\}^{1/2}$ is positive. Thus,

$$H(X) = 1/2\pi i \int_{\mathscr{L}_2} \frac{h(s)\, ds}{\{(x-s)^2 + \rho^2\}^{1/2}} \equiv N(X). \tag{51}$$

Combining this result with Theorem 1 leads us to

THEOREM 5. *Let h be analytic on ω, $\mathscr{L}$ be a contour in ω intersecting the real axis in two points, and ω_s be an axiconvex set enclosed by $\mathscr{L}$ such that $\partial\omega_s \cap \mathscr{L} = \phi$. Then on Ω,*

$$N(X) = 1/2\pi i \int_{\mathscr{L}} \frac{h(s)\, ds}{\{(x-s)^2 + \rho^2\}^{1/2}} \tag{52}$$

with $N(X) \in h^*(\sigma_0)$ for all circles $X \prec X_0$ where the branch of the integrand is chosen so that $\{x^2 + \rho^2\}^{1/2}$ is positive.

We apply this result in

COROLLARY 5.1. *The Newtonian potential*

$$N(X) = \sum_{n=0}^{m} a_{m-n} \int_{\mathscr{L}} \frac{s^n\, ds}{\{(x-s)^2 + \rho^2\}^{1/2}}, \tag{53}$$

has no zeros in the double cone

$$0 < \rho \leqslant \pm\, x \tan(\pi/2m) - \sec(\pi/2m) \tag{54}$$

if $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_m > 0$.

Proof. The associate of $N(X)$ is identified as $h(\zeta) = \sum_{n=0}^{m} a_{m-n} \zeta^n$. The Eneström–Kakeya Theorem [5, p. 136] guarantees that the zeros of the reciprocal polynomial

$$g(\zeta) = \zeta^m h(\zeta^{-1}) \tag{55}$$

are in $|\zeta| > 1$ so that $h(\zeta_0) = 0$ only if $|\zeta_0| \leqslant 1$. We then use previous reasoning in Eq. (51) to reach our conclusion.

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